von Neumann Regular and Related Elements in Commutative Rings

Ayman Badawi
Department of Mathematics & Statistics, The American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates
Co-Author: David F. Anderson
Department of Mathematics, The University of Tennessee, Knoxville, USA

Dedicated to Syed Tariq Rizvi

Let R be a commutative ring with nonzero identity. we define $a \in R$ to be a *von Neumann regular element* of R (or just *von Neumann regular*) if $a^2x = a$ for some $x \in R$. Similarly, we define $a \in R$ to be a π-*regular element* of R (or just π-*regular*) if $a^{2n}x = a^n$ for some $x \in R$ and integer $n \geq 1$. Let $Idem(R) = \{ a \in R \mid a^2 = a \}$, $vnr(R) = \{ a \in R \mid a$ is von Neumann regular $\}$, and $\pi\text{-}r(R) = \{ a \in R \mid a$ is π-regular $\}$. Thus $Idem(R) \subseteq vnr(R) \subseteq \pi\text{-}r(R)$ and R is a Boolean (resp., von Neumann regular, π-regular) ring if and only if $Idem(R) = R$ (resp., $vnr(R) = R$, $\pi\text{-}r(R) = R$).

Theorem

Let R be a commutative ring. Then the following statements are equivalent for $a \in R$.
(1) $a \in vnr(R)$.
(2) $a^2u = a$ *for some* $u \in U(R)$.
(3) $a = ue$ *for some* $u \in U(R)$ *and* $e \in Idem(R)$.
(4) $ab = 0$ *for some* $b \in vnr(R) \setminus \{a\}$ *with* $a + b \in U(R)$.
(5) $ab = 0$ *for some* $b \in R$ *with* $a + b \in U(R)$.

Let $a \in vnr(R)$. Then $a^2x = a$ for some $x \in R$. Note that x need not be unique since we may replace x by any $y \in x + ann(a^2)$. The following result is well known for von Neumann regular rings.

Theorem

Let R be a commutative ring and $a \in vnr(R)$. Then there is a unique $x \in R$ with $a^2x = a$ and $x^2a = x$.

Since $vnr(R) \cap nil(R) = \{0\}$, it is natural to ask when $R = vnr(R) \cup nil(R)$, i.e., when is every non-nilpotent element of R von Neumann regular?

Theorem

Let R be a commutative ring.
(1) $R = vnr(R) \cup nil(R)$ if and only if either R is von Neumann regular or R is quasilocal with maximal ideal $nil(R)$. In particular, if $R = vnr(R) \cup nil(R)$, then R is a π-regular ring.
(2) $R = vnr(R) \cup Z(R)$ if and only if $T(R) = R$.

We next show that if $\{0\} \subsetneq Z(R) \subseteq vnr(R)$, then R is von Neumann regular. One consequence of the next result is that to check if a non-domain R is von Neumann regular, we need only show that each zero-divisor of R is von Neumann regular.

Theorem

Let R be a commmutative ring with $\{0\} \subsetneq Z(R)$. Then $Z(R) \subseteq vnr(R)$ if and only if R is von Neumann regular.

Remark

D. D. Anderson and V. P. Camillo proved that $R = U(R) \cup Idem(R)$ if and only if R is a Boolean ring,

Theorem

Let R be a commutative ring. Then $R = Idem(R) \cup nil(R)$ if and only if R is Boolean.

Theorem

Let R be a commmutative ring with $\{0\} \subsetneq Z(R)$. Then $Z(R) \subseteq Idem(R)$ if and only if R is Boolean.

It seems natural to conjecture that $R = Idem(R) \cup Z(R)$ if and only if R is a Boolean ring. We next give some evidence to support this conjecture.

Theorem

Let R be a commutative ring.
(1) If $R = Idem(R) \cup Z(R)$, then $U(R) = \{1\}$, $char(R) = 2, nil(R) = \{0\}$, $J(R) = \{0\}$, and $T(R) = R$.
(2) If either $dim(R) = 0$ or R has only a finite number of maximal ideals, then $R = Idem(R) \cup Z(R)$ if and only if R is Boolean.

It is well known that if R is a commutative von Neumann regular ring with $2 \in U(R)$, then every element of R is the sum of two units of R. G. Ehrlich proved that if $aua = a$ for some $u \in U(R)$, then a is the sum of two units of R. So this result extends to $vnr(R)$.

Theorem

([15]) Let R be a commutative ring with $2 \in U(R)$. Then every $a \in vnr(R)$ is the sum of two units of R.

Theorem

Let R be a commutative ring with $2 \in U(R)$. Then the following statements are equivalent.
(1) $vnr(R)$ is a subring of R.
(2) The sum of any four units of R is a von Neumann regular element of R.
(3) Let $u, v, k, m \in U(R)$ with $k^2 = m^2 = 1$. Then $u(1+k) + v(1+m) \in vnr(R)$.

Recall that for a commutative ring R, we let
$\pi\text{-}r(R) = \{ a \in R \mid a^{2n}x = a^n$ for some $x \in R$ and integer

$n \geq 1\}$ be the set of π-regular elements of R. Thus R is π-regular if and only if π-$r(R) = R$, if and only if $\dim(R) = 0$.

Theorem

Let R be a commutative ring. Then the following statements are equivalent for $a \in R$.

(1) $a \in \pi\text{-}r(R)$.

(2) $a^n \in vnr(R)$ *for some integer* $n \geq 1$.

(3) $a^n = ue$ *for some* $u \in U(R), e \in Idem(R)$, *and integer* $n \geq 1$.

(4) $a = b + w$ *for some* $b \in vnr(R)$ *and* $w \in nil(R)$.

(5) $a = ue + w$ *for some* $u \in U(R), e \in Idem(R)$, *and* $w \in nil(R)$.

(6) $a + nil(R) \in vnr(R/nil(R))$.

(7) $a^n b = 0$ *for some* $b \in R$ *and integer* $n \geq 1$ *with* $a^n + b \in U(R)$.

(8) $ab \in nil(R)$ *for some* $b \in R$ *with* $a + b \in U(R)$.

It is natural to ask when π-$r(R) = vnr(R) \cup nil(R)$.

Theorem

Let R be a commutative ring.
(1) π-$r(R) = vnr(R) \cup nil(R)$ if and only if either $Idem(R) = \{0, 1\}$ or $nil(R) = \{0\}$.
(2) $R = \pi$-$r(R) \cup Z(R)$ if and only if $T(R) = R$.

In the follwing result we show if a ring R with $nil(R) \subsetneq Z(R)$ is π-regular, we only need check that the zero-divisors of R are all π-regular.

Theorem

Let R be a commutative ring with $nil(R) \subsetneq Z(R)$. Then $Z(R) \subseteq \pi$-$r(R)$ if and only if R is π-regular.

Recall that $nil(R)$ is of *bounded index n* if n is the least positive integer such that $w^n = 0$ for every $w \in nil(R)$. A commutative ring R is said to be of *bounded index n* if n is the least positive integer such that $a^n \in vnr(R)$ for every $a \in \pi\text{-}r(R)$. Note that a von Neumann regular ring is of bounded index 1.

Theorem

Let R be a commutative ring and n a positive integer. Then R is of bounded index n if and only if $nil(R)$ is of bounded index n.

Recall (M. Contessa),)that a commutative ring R is a *von Neumann local ring* if either $a \in vnr(R)$ or $1 - a \in vnr(R)$ for every $a \in R$. This concept have been further studied by E. Abu Osba, M. Henrikson, O. Alkam, and F. A. Smith We define $vnl(R) = \{\, a \in R \mid a \in vnr(R) \text{ or } 1 - a \in vnr(R) \,\}$ to be the set of von Neumann local elements of R. Thus R is a von Neumann local ring if and only if $vnl(R) = R$.

Theorem

Let R be a commutative rings. Then
(1) $vnl(R) = vnr(R) \cup (1 + vnr(R)) = \{0,1\} + vnr(R)$. In particular, $\{0,1\} + U(R) = U(R) \cup (1 + U(R)) \subseteq vnl(R)$.
(2) Let $a \in R$. Then $a \in vnl(R)$ if and only if there is a $u \in U(R)$ and $e \in Idem(R)$ such that either $a = ue$ or $a = 1 + ue$.
(3) $nil(R) \subseteq J(R) \subseteq vnl(R)$. Thus $U(R) \cup J(R) \subseteq vnl(R)$.
(4) $vnl(R) = U(R) \cup (1 + U(R))$ if and only if $Idem(R) = \{0,1\}$. In particular, $vnl(R) = U(R) \cup (1 + U(R))$ when R is either an integral domain or quasilocal (note that $vnl(R) = R$ when R is qusailocal).

Recall ()W. K. Nicholson) that a commutative ring R is a clean ring if for every $a \in R$, $a = u + e$ for some $u \in U(R)$ and $e \in Idem(R)$. We define $cln(R) = \{ a \in R \mid a = u + e$ for some $u \in U(R)$ and $e \in Idem(R)\} = U(R) + Idem(R)$ to be the set of clean elements of R. Thus R is a clean ring if and only if $cln(R) = R$.

Theorem

Let R be commutative ring. Then
(1) $Idem(R) \subseteq vnr(R) \subseteq vnl(R) \subseteq cln(R)$. *In particular, a Boolean ring, a von Neumann regular ring, or a von Neumann local ring is a clean ring.*
(2) $vnr(R) \subseteq \pi\text{-}r(R) \subseteq cln(R)$. *In particular, a π-regular ring is a clean ring.*
(3) $U(R) \cup J(R) \subseteq U(R) \cup (1 + U(R)) \subseteq cln(R)$.
(4) If $Idem(R) = \{0, 1\}$, *then* $cln(R) = vnl(R)$. *In particular,* $cln(R) = vnl(R)$ *when R is either an integral domain or quasilocal (note that* $cln(R) = vnl(R) = R$ *when R is quasilocal).*
(7) If $2 \in U(R)$, *then every* $a \in cln(R)$ *is the sum of three units of R.*
(8) If $vnl(R)$ *is multiplicatively closed, then* $cln(R) = vnl(R)$.

Theorem

Let R be a commutative ring, and consider the following statements.

(a) $vnl(R) = U(R) \cup nil(R)$.
(b) $cln(R) = U(R) \cup nil(R)$.
(c) $vnl(R) = vnr(R) \cup nil(R)$.
(d) $cln(R) = vnr(R) \cup nil(R)$.

Then (1) (a) $\Leftrightarrow$ (b), (c) $\Leftrightarrow$ (d), *and* (a) $\Rightarrow$ (c).
(2) *If any of the four statements holds, then* $\pi\text{-}r(R) = vnl(R) = cln(R)$.
(3) *If (a) or (b) holds, then* $Idem(R) = \{0, 1\}$.
(4) *If (c) or (d) holds, then either* $Idem(R) = \{0, 1\}$ *or* $nil(R) = \{0\}$.

To get the full paper free of charge, click here
http://ayman-badawi.com/Publication.html

- E. Abu Osba, M. Henrikson, and O. Alkam, *Combing local and von Neumann regular rings*, Comm. Algebra 32(2004), 2639-2653.
- E. Abu Osba, M. Henrikson, O. Alkam, and F. A. Smith, *The maximal regular ideal of some commutative rings*, Comment. Math. Univ. Carolin. 47(2006), 1-10.
- M. S. Ahn and D. D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mountain J. Math. 30(2006), 783-798.
- D. D. Anderson and M. Winders, *Idealizaion of a module*, J. Commutative Algebra 1(2009), 3-56.

- D. D. Anderson and V. P. Camillo, *Commutative rings whose elements are a sum of a unit and an idempotent*, Comm. Algebra 30(2002), 3327-3336.
- D. F. Anderson, M. C. Axtell, and J. A. Stickles, *Zero-divisor graphs in commutative rings*, in *Recent Developments in Commutative Algebra, Noetherian and Non-Noetherian Perspectives* (M. Fontana, S.-E. Kabbaj, B. Olberding, I. Swanson, Eds.), Springer-Verlag, New York, 2011, 23-45.
- D. F. Anderson and A. Badawi, *Divisibility conditions in commutative rings with zero-divisors*, Comm. Algebra 30(2002), 4031-4047.

D. F. Anderson and A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra 36(2008), 3073-3092.

D. F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebra*, J. Pure Appl. Algebra 180(2003), 221-241.

D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217(1999), 434-447.

M. Contessa, *On certain classes of MP rings*, Comm. Algebra 12(1984), 1447-1469.

M. D'Anna and and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. 6(2007), 443-459.

- M. D'Anna and and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. 45(2007), 241-252.
- R. Diestel, *Graph Theory*, Springer, New York, 1997.
- G. Ehrlich, *Unit-regular rings*, Portugal. Math. 27(1968), 209-212.
- J. W. Fisher and R. L. Synder, *Rings generated by their units*, J. Algebra 42(1976), 363-368.
- R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York/Basel, 1972.
- J. Han and W. K. Nicholson, *Extensions of clean rings*, Comm. Algebra 29(2001), 2589-2595.

J. A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York/Basel, 1988.

K. N. Kyun and L. Yang *On right quasi-duo rings which are π-regular*, Bull. Korean Math. Soc. 37(2000), 217-227.

J. D. LaGrange, *Complemented zero-divisor graphs and Boolean rings*, J. Algebra 315(2007), 600-611.

J. D. LaGrange, *On realizing zero-divisor graphs*, Comm. Algebra 36(2008), 4509-4520.

J. D. LaGrange, *Characterizations of three classes of zero-divisor graphs*, Canad. Math. Bull., to appear.

T. Y. Lam, *Exercises in Classical Ring Theory*, Springer, New York, 2003.

J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham/Toronto/London, 1966.

R. Levy and J. Shapiro, *The zero-divisor graph of von Neumann regular rings*, Comm. Algebra 30(2002), 745-750.

W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. 229(1977), 269-278.

R. Raphael, *Some remarks on regular and strongly regular rings*. Canad. Math. Bull. 17(1975), 709-712.